

RELATIVE DERIVED CATEGORY WITH RESPECT TO A SUBCATEGORY

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Abstract

The notion of relative derived category with respect to a subcategory is introduced. A triangle-equivalence, which extends a theorem of Gao and Zhang [Gorenstein derived categories, *J. Algebra* **323** (2010) 2041-2057] to the bounded below case, is obtained. Moreover, we interpret the relative derived functor $\text{Ext}_{\mathcal{KA}}(-, -)$ as the morphisms in such derived category and give two applications.

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1. Introduction

For an abelian category \mathcal{A} with enough projective objects, the Gorenstein (projective) derived category, which makes Gorenstein projective quasi-isomorphisms become isomorphisms, was introduced by Gao and Zhang [10] to close a gap of the corresponding version of derived category in so-called Gorenstein homological algebra. In particular, they established the triangle-equivalence

$$\mathbf{K}^b(\mathcal{G}(\mathcal{P})) \cong \mathbf{D}_{\mathcal{G}(\mathcal{P})}^b(\text{res } \widehat{\mathcal{G}(\mathcal{P})}) \quad (\dagger)$$

where $\text{res } \widehat{\mathcal{G}(\mathcal{P})}$ denotes the full subcategory of objects in \mathcal{A} with finite Gorenstein projective dimension (see [10, Theorem 3.6]). Moreover, they showed, as one of the advantages of such relative derived category, the relative derived functor

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$\text{Ext}_{\mathcal{G}(\mathcal{P})}(-, -)$, which is derived from $\text{Hom}_{\mathcal{A}}(-, -)$ using proper $\mathcal{G}(\mathcal{P})$ -resolutions of the first variable, can be interpreted as the morphisms in $\mathbf{D}_{\mathcal{G}(\mathcal{P})}^b(\mathcal{A})$ (see [10, Theorem 3.12]).

Let \mathcal{X} and \mathcal{S} be subcategories of \mathcal{A} with \mathcal{S} closed under direct summands. We introduce in the paper the notion of relative derived category with respect to \mathcal{X} , $\mathbf{D}_{\mathcal{X}}^*(\mathcal{S})$ for $*$ $\in \{\text{blank}, -, b\}$, which is defined to be the Verdier quotient of the homotopy category $\mathbf{K}^*(\mathcal{S})$ with respect to the thick triangulated subcategory of all \mathcal{X} -acyclic complexes in $\mathbf{K}^*(\mathcal{S})$ (see Definition 3.4). Then we obtain in section 3 the following triangle-equivalence, which extends (\dagger) to the bounded below case when we specially take $\mathcal{X} = \mathcal{G}(\mathcal{P})$ (see Theorem 3.10). Also, it can be compared to [5, Proposition 3.5] (see Example 3.11).

Theorem II *Let \mathcal{X} be a subcategory of \mathcal{A} . Assume that \mathcal{X} is exact and has an injective cogenerator. Then there exists a triangle-equivalence $\mathbf{K}^-(\mathcal{X}) \cong \mathbf{D}_{\mathcal{X}}^-(\text{res } \widehat{\mathcal{X}})$.*

The study of relative homological algebra goes back to Butler and Horrocks [2], and Eilenberg and Moore [7]. It was reinvigorated recently by a number of authors (see, for example, Enochs and Jenda [6], and Avramov and Martsinkovsky [1]). Assume that M and N are objects in \mathcal{A} such that M admits a proper \mathcal{X} -resolution \mathbf{X} . According to [20], the relative cohomology groups $\text{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N)$ is defined as $H^n(\text{Hom}_{\mathcal{A}}(\mathbf{X}, N))$. In section 4, inspired by [10, Theorem 3.12], we give the following new characterization of $\text{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N)$ (see Theorem 4.2).

Theorem III *Let \mathcal{X} be a subcategory of \mathcal{A} and M, N objects in \mathcal{A} . Assume that M admits a proper \mathcal{X} -resolution. Then $\text{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N) = \text{Hom}_{\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})}(M, N[n])$.*

Moreover, as an application of Theorem III, we give in derived category a new proof for the existence of the Avramov-Martsinkovsky type exact sequence appeared in [22, Theorem B] (see Corollary 4.6).

In the next section, we mainly review some definitions and notation. It should be pointed out that many of them were used by Sather-Wagstaff et al. [19-22].

2. Preliminaries

Throughout this paper, \mathcal{A} denotes an abelian category with enough projective objects. The term *subcategory* stands for a full additive subcategory of \mathcal{A} closed under isomorphisms. Following [21], a subcategory of \mathcal{A} is called *exact* if it is closed under extensions and direct summands. We write $\mathcal{P} = \mathcal{P}(\mathcal{A})$ for the subcategory of projective objects in \mathcal{A} .

Let \mathcal{X}, \mathcal{Y} and \mathcal{W} be subcategories of \mathcal{A} such that $\mathcal{W} \subseteq \mathcal{X}$. We write $\mathcal{X} \perp \mathcal{Y}$ in case $\text{Ext}_{\mathcal{A}}^{\geq 1}(X, Y) = 0$ for each object $X \in \mathcal{X}$ and each object $Y \in \mathcal{Y}$. When $\mathcal{X} = \{M\}$,

we use the notation $M \perp \mathcal{Y}$ instead of $\{M\} \perp \mathcal{Y}$. There is an analogue $\mathcal{X} \perp M$. According to [21], \mathcal{W} is said to be a *cogenerator* for \mathcal{X} if for each object $X \in \mathcal{X}$, there exists a short exact sequence $0 \rightarrow X \rightarrow W \rightarrow X' \rightarrow 0$ such that $W \in \mathcal{W}$ and $X' \in \mathcal{X}$. \mathcal{W} is called an *injective cogenerator* for \mathcal{X} if \mathcal{W} is a cogenerator for \mathcal{X} and $\mathcal{X} \perp \mathcal{W}$. *Generator* and *projective generator* can be defined dually.

A *complex* \mathbf{X} is often displayed as a sequence of objects in \mathcal{A}

$$\cdots \longrightarrow X^{-1} \xrightarrow{\delta_{\mathbf{X}}^{-1}} X^0 \xrightarrow{\delta_{\mathbf{X}}^0} X^1 \longrightarrow \cdots$$

with $\delta_{\mathbf{X}}^{n+1}\delta_{\mathbf{X}}^n = 0$ for all $n \in \mathbb{Z}$. The n th *homology* of the complex \mathbf{X} is defined as $H^n(\mathbf{X}) = \text{Ker}(\delta_{\mathbf{X}}^n)/\text{Im}(\delta_{\mathbf{X}}^{n-1})$. We identify M with the complex $\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$, where M is in degree zero and 0 elsewhere. For an integer n , $\mathbf{X}[n]$ denotes the complex \mathbf{X} shifting n degree, that is, $X[n]^m = X^{n+m}$ and $\delta_{\mathbf{X}[n]}^m = (-1)^n \delta_{\mathbf{X}}^{n+m}$. Given two complexes \mathbf{X} and \mathbf{Y} , the complex $\text{Hom}_{\mathcal{A}}(\mathbf{X}, \mathbf{Y})$ is defined with $\text{Hom}_{\mathcal{A}}(\mathbf{X}, \mathbf{Y})^n = \prod_{k \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(X^k, Y^{k+n})$, and with differential $\delta^n((f^k)_{k \in \mathbb{Z}}) = (\delta_{\mathbf{Y}}^{k+n} f^k - (-1)^n f^{k+1} \delta_{\mathbf{X}}^k)_{k \in \mathbb{Z}}$ for $(f^k)_{k \in \mathbb{Z}} \in \text{Hom}_{\mathcal{A}}(\mathbf{X}, \mathbf{Y})^n$.

A *morphism* $f : \mathbf{X} \rightarrow \mathbf{Y}$ of complexes is a family of morphisms $f = (f^n : X^n \rightarrow Y^n)_{n \in \mathbb{Z}}$ of objects in \mathcal{A} satisfying $\delta_{\mathbf{Y}}^n f^n = f^{n+1} \delta_{\mathbf{X}}^n$ for all $n \in \mathbb{Z}$. Morphisms $f, g : \mathbf{X} \rightarrow \mathbf{Y}$ are called *homotopic*, denoted by $f \sim g$, if there exists a family of morphisms $(s^n : X^n \rightarrow Y^{n-1})_{n \in \mathbb{Z}}$ of objects in \mathcal{A} satisfying $f^n - g^n = \delta_{\mathbf{Y}}^{n-1} s^n + s^{n+1} \delta_{\mathbf{X}}^n$ for all $n \in \mathbb{Z}$. A *quasi-isomorphism* is a morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$ with $H^n(f) : H^n(\mathbf{X}) \rightarrow H^n(\mathbf{Y})$ bijective for all $n \in \mathbb{Z}$.

A complex \mathbf{X} is called an \mathcal{X} -*resolution* of M if $X^i \in \mathcal{X}$ for all $i \leq 0$, $X^i = 0$ for all $i > 0$, $H^i(\mathbf{X}) = 0$ for all $i < 0$ and $H^0(\mathbf{X}) \cong M$. In this case, the associated exact sequence

$$\cdots \rightarrow X^{-1} \rightarrow X^0 \rightarrow M \rightarrow 0$$

is denoted by \mathbf{X}^+ . Sometimes we call the quasi-isomorphism $\mathbf{X} \xrightarrow{\sim} M$ an \mathcal{X} -resolution of M . If \mathbf{X}^+ is $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$ -exact, we say that \mathbf{X} is a *proper* \mathcal{X} -resolution of M , and let $\text{res } \tilde{\mathcal{X}}$ denote the subcategory of objects in \mathcal{A} admitting a proper \mathcal{X} -resolution.

The \mathcal{X} -*projective dimension* of M is the least non-negative n such that there exists an exact sequence $0 \rightarrow X^{-n} \rightarrow \cdots \rightarrow X^{-1} \rightarrow X^0 \rightarrow M \rightarrow 0$, where each $X^{-i} \in \mathcal{X}$. In this case, we write $\mathcal{X}\text{-pd}(M) = n$. If no such n exists, we write $\mathcal{X}\text{-pd}(M) = \infty$. The subcategory of objects in \mathcal{A} with finite \mathcal{X} -projective dimension is denoted by $\widehat{\text{res } \tilde{\mathcal{X}}}$.

For $* \in \{\text{blank}, -, b\}$, $\mathbf{K}^*(\mathcal{A})$ and $\mathbf{D}^*(\mathcal{A})$ stand for the corresponding homotopy category and derived category of \mathcal{A} , respectively. We will often use the standard formula $\text{Hom}_{\mathbf{K}^*(\mathcal{A})}(\mathbf{X}, \mathbf{Y}[n]) = H^n(\text{Hom}_{\mathcal{A}}(\mathbf{X}, \mathbf{Y})) = \text{Hom}_{\mathbf{K}^*(\mathcal{A})}(\mathbf{X}[-n], \mathbf{Y})$.

Let \mathcal{B} be a triangulated subcategory of a triangulated category \mathcal{K} , and let Σ be the compatible multiplicative system determined by \mathcal{B} . In the Verdier quotient [17, Chapter 2] $\mathcal{K}/\mathcal{B} = \Sigma^{-1}\mathcal{K}$, each morphism $f : X \rightarrow Y$ is given by an equivalence class of right fractions a/s presented by $X \xleftarrow{s} Z \xrightarrow{a} Y$, where the double arrow means $s \in \Sigma$.

3. Relative Derived Categories

In what follows, we always assume that \mathcal{X} and \mathcal{S} are subcategories of \mathcal{A} with \mathcal{S} closed under direct summands. We begin with the following definition.

Definition 3.1. A complex \mathbf{S} is called \mathcal{X} -acyclic if $\mathrm{Hom}_{\mathcal{A}}(X, \mathbf{S})$ is acyclic for each object $X \in \mathcal{X}$. A morphism $f : \mathbf{S} \rightarrow \mathbf{T}$ of complexes is called an \mathcal{X} -quasi-isomorphism if $\mathrm{Hom}_{\mathcal{A}}(X, f)$ is a quasi-isomorphism for each object $X \in \mathcal{X}$.

Let \mathcal{T} be a subcategory of \mathcal{A} . For $* \in \{\text{blank}, -, b\}$, we denote by $\mathbf{K}^*(\mathcal{T})$ the homotopy category with each complex constructed by objects in \mathcal{T} and by $\mathbf{K}_{\mathcal{X}}^*(\mathcal{T})$ the subcategory of $\mathbf{K}^*(\mathcal{T})$ consisting of all \mathcal{X} -acyclic complexes.

Fact 3.2. *By virtue of [4, Lemma 2.4], a complex \mathbf{S} is \mathcal{X} -acyclic if and only if $\mathrm{Hom}_{\mathcal{A}}(\mathbf{D}, \mathbf{S})$ is acyclic for each complex $\mathbf{D} \in \mathbf{K}^-(\mathcal{X})$. Moreover, it follows from [4, Proposition 2.6] that a morphism $f : \mathbf{S} \rightarrow \mathbf{T}$ of complexes is an \mathcal{X} -quasi-isomorphism if and only if $\mathrm{Hom}_{\mathcal{A}}(\mathbf{D}, f)$ is a quasi-isomorphism for each complex $\mathbf{D} \in \mathbf{K}^-(\mathcal{X})$.*

Recall from [11] that a full triangulated subcategory \mathcal{C} of a triangulated category \mathcal{D} is said to be *thick* if it satisfies the following condition: assume that a morphism $f : X \rightarrow Y$ in \mathcal{D} can be factored through an object from \mathcal{C} , and enters a distinguished triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1]$ with Z in \mathcal{C} , then X and Y are objects in \mathcal{C} . A standard example of thick subcategory is the subcategory of all acyclic complexes in $\mathbf{K}(\mathcal{A})$, that is, $\mathbf{K}_{\mathcal{P}}^*(\mathcal{A})$.

The following result depends on an important characterization of thick subcategories due to Rickard, called *Rickard's criterion*: a full triangulated subcategory \mathcal{C} of a triangulated category \mathcal{D} is thick if and only if every direct summand of an object of \mathcal{C} is in \mathcal{C} (see [18, Proposition 1.3] or [16, Criterion 1.3]).

Lemma 3.3. *For $* \in \{\text{blank}, -, b\}$, $\mathbf{K}_{\mathcal{X}}^*(\mathcal{S})$ is a thick subcategory of $\mathbf{K}^*(\mathcal{S})$.*

Proof. By Rickard's criterion, it suffices to show that $\mathbf{K}_{\mathcal{X}}^*(\mathcal{S})$ is a full triangulated subcategory of $\mathbf{K}^*(\mathcal{S})$ and it is closed under direct summands. This is clear. \square

Note that a morphism $f : \mathbf{S} \rightarrow \mathbf{T}$ of complexes is an \mathcal{X} -quasi-isomorphism if and only if its mapping cone $\text{cone}(f)$ is \mathcal{X} -acyclic. Then, the collection of all \mathcal{X} -quasi-isomorphisms in $\mathbf{K}^*(\mathcal{S})$, denoted by $\Sigma_{\mathcal{X}}^{\mathcal{S}}$, is a saturated multiplicative system corresponding to the subcategory $\mathbf{K}_{\mathcal{X}}^*(\mathcal{S})$.

Definition 3.4. For $*$ $\in \{\text{blank}, -, b\}$, the *relative derived category* $\mathbf{D}_{\mathcal{X}}^*(\mathcal{S})$ is defined to be the Verdier quotient of $\mathbf{K}^*(\mathcal{S})$, that is,

$$\mathbf{D}_{\mathcal{X}}^*(\mathcal{S}) := \mathbf{K}^*(\mathcal{S}) / \mathbf{K}_{\mathcal{X}}^*(\mathcal{S}) = \Sigma_{\mathcal{X}}^{-\mathcal{S}} \mathbf{K}^*(\mathcal{S}).$$

Note that $\mathbf{D}_{\mathcal{X}}^*(\mathcal{S})$ is the derived category in sense of Neeman [16], of the exact category $(\mathcal{S}, \mathcal{E}_{\mathcal{X}}^{\mathcal{S}})$, where $\mathcal{E}_{\mathcal{X}}^{\mathcal{S}}$ consists of all short \mathcal{X} -acyclic sequences in \mathcal{S} .

The next result will be used in the proof of Lemma 3.6.

Lemma 3.5. *Let \mathbf{S} be a complex. Assume that there is a complex $\mathbf{D} \in \mathbf{K}^-(\mathcal{X})$ and an \mathcal{X} -quasi-isomorphism $f : \mathbf{S} \rightarrow \mathbf{D}$. Then there exists a morphism $g : \mathbf{D} \rightarrow \mathbf{S}$ such that fg is homotopic to $\text{Id}_{\mathbf{D}}$.*

Proof. In view of Fact 3.2, $\text{Hom}_{\mathcal{A}}(\mathbf{D}, f) : \text{Hom}_{\mathcal{A}}(\mathbf{D}, \mathbf{S}) \rightarrow \text{Hom}_{\mathcal{A}}(\mathbf{D}, \mathbf{D})$ is a quasi-isomorphism. Then by [1, (1.1)], for the morphism $\text{Id}_{\mathbf{D}}$, there is a morphism $g : \mathbf{D} \rightarrow \mathbf{S}$ such that $fg \sim \text{Id}_{\mathbf{D}}$. \square

By a slight modification of the proof of [10, Proposition 2.8], we can get the following result, which makes the morphisms in $\mathbf{D}_{\mathcal{X}}^-(\mathcal{S})$ easier to understand.

Lemma 3.6. *Let \mathbf{D} be a complex in $\mathbf{K}^-(\mathcal{X})$ and \mathbf{S} a complex in $\mathbf{K}^-(\mathcal{S})$. Assume $\mathcal{X} \subseteq \mathcal{S}$. Then $\varphi : f \rightarrow f/\text{Id}_{\mathbf{D}}$ gives an isomorphism of abelian groups $\text{Hom}_{\mathbf{K}^-(\mathcal{S})}(\mathbf{D}, \mathbf{S}) \cong \text{Hom}_{\mathbf{D}_{\mathcal{X}}^-(\mathcal{S})}(\mathbf{D}, \mathbf{S})$.*

Proof. If $f/\text{Id}_{\mathbf{D}} = 0$, then by the calculus of right fractions there is an \mathcal{X} -quasi-isomorphism $s : \mathbf{Y} \rightarrow \mathbf{D}$ for some complex \mathbf{Y} such that $fs \sim 0$. It follows from Lemma 3.5 that there is a morphism $g : \mathbf{D} \rightarrow \mathbf{Y}$ such that $sg \sim \text{Id}_{\mathbf{D}}$. Thus, $f \sim fsg \sim 0$. Moreover, for each $f/s \in \text{Hom}_{\mathbf{D}_{\mathcal{X}}^-(\mathcal{S})}(\mathbf{D}, \mathbf{S})$ presented by $\mathbf{D} \xleftarrow{s} \mathbf{Y} \xrightarrow{f} \mathbf{S}$, by Lemma 3.5 there is a morphism $g : \mathbf{D} \rightarrow \mathbf{Y}$ such that $sg \sim \text{Id}_{\mathbf{D}}$. This implies that $f/s = fg/\text{Id}_{\mathbf{D}} = \varphi(fg)$. Thus, φ is an isomorphism, as desired. \square

The next result is given in service of Lemma 3.8.

Lemma 3.7. *Let $\mathbb{L} = 0 \rightarrow \text{Ker}(g) \rightarrow \mathbf{M} \xrightarrow{g} \mathbf{N} \rightarrow 0$ be a short exact sequence of complexes such that $\text{Ker}(g)$ is \mathcal{X} -acyclic and \mathbb{L} remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$, i.e., the sequence*

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{X}, \text{Ker}(g)) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathbf{M}) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{X}, \mathbf{N}) \rightarrow 0 \quad (\ddagger)$$

of complexes is exact for each object $X \in \mathcal{X}$. Then for each complex $\mathbf{D} \in \mathbf{K}^-(\mathcal{X})$ and each morphism $\alpha : \mathbf{D} \rightarrow \mathbf{N}$, there exists a morphism $\beta : \mathbf{D} \rightarrow \mathbf{M}$ such that the following diagram

$$\begin{array}{ccccccc} & & & & \mathbf{D} & & \\ & & & \nearrow \beta & \downarrow \alpha & & \\ 0 & \longrightarrow & \text{Ker}(g) & \longrightarrow & \mathbf{M} & \xrightarrow{g} & \mathbf{N} \longrightarrow 0 \end{array}$$

commutes.

Proof. Without loss of generality, we may assume

$$\mathbf{D} = \cdots \rightarrow D^{-2} \rightarrow D^{-1} \rightarrow D^0 \rightarrow 0 \rightarrow \cdots.$$

To complete this proof, we need to construct a morphism $\beta = (\beta^k)_{k \in \mathbb{Z}} : \mathbf{D} \rightarrow \mathbf{M}$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D^{-2} & \xrightarrow{\delta_{\mathbf{D}}^{-2}} & D^{-1} & \xrightarrow{\delta_{\mathbf{D}}^{-1}} & D^0 \longrightarrow 0 \longrightarrow \cdots \\ & & \downarrow \beta^{-2} & \searrow \alpha^{-2} & \downarrow \beta^{-1} & \searrow \alpha^{-1} & \downarrow \beta^0 \searrow \alpha^0 \\ & & \cdots & \longrightarrow & N^{-2} & \longrightarrow & N^{-1} \longrightarrow \cdots \\ & & \downarrow & \nearrow g^{-2} & \downarrow & \nearrow g^{-1} & \downarrow & \nearrow g^0 \\ \cdots & \longrightarrow & M^{-2} & \xrightarrow{\delta_{\mathbf{M}}^{-2}} & M^{-1} & \xrightarrow{\delta_{\mathbf{M}}^{-1}} & M^0 \xrightarrow{\delta_{\mathbf{M}}^0} M^1 \longrightarrow \cdots \end{array}$$

commutes.

Let $\beta^k = 0$ for $k \geq 1$. Since (\dagger) is exact, $0 \rightarrow \text{Ker}(g^i) \rightarrow M^i \xrightarrow{g^i} N^i \rightarrow 0$ is $\text{Hom}_{\mathcal{A}}(X, -)$ -exact for each $i \in \mathbb{Z}$ and each object $X \in \mathcal{X}$. Hence, there exists $\gamma^i : D^i \rightarrow M^i$ such that $g^i \gamma^i = \alpha^i$ for each $i \in \mathbb{Z}$. Note that

$$\text{Ker}(g) = \cdots \rightarrow \text{Ker}(g^{-1}) \xrightarrow{\delta_{\mathbf{M}}^{-1}} \text{Ker}(g^0) \xrightarrow{\delta_{\mathbf{M}}^0} \text{Ker}(g^1) \xrightarrow{\delta_{\mathbf{M}}^1} \text{Ker}(g^2) \rightarrow \cdots$$

is \mathcal{X} -acyclic. As $D^0 \in \mathcal{X}$, we have an exact sequence

$$\cdots \rightarrow \text{Hom}_{\mathcal{A}}(D^0, \text{Ker}(g^0)) \rightarrow \text{Hom}_{\mathcal{A}}(D^0, \text{Ker}(g^1)) \rightarrow \text{Hom}_{\mathcal{A}}(D^0, \text{Ker}(g^2)) \rightarrow \cdots.$$

Since $g^1 \delta_{\mathbf{M}}^0 \gamma^0 = \delta_{\mathbf{N}}^0 g^0 \gamma^0 = \delta_{\mathbf{N}}^0 \alpha^0 = 0$, $-\delta_{\mathbf{M}}^0 \gamma^0 \in \text{Hom}_{\mathcal{A}}(D^0, \text{Ker}(g^1))$. Now $\delta_{\mathbf{M}}^1 \delta_{\mathbf{M}}^0 \gamma^0 = 0$, and thus there exists $\mu^0 : D^0 \rightarrow \text{Ker}(g^0)$ such that $\delta_{\mathbf{M}}^0 \mu^0 = -\delta_{\mathbf{M}}^0 \gamma^0$. Let $\beta^0 = \mu^0 + \gamma^0$. Then $\delta_{\mathbf{M}}^0 \beta^0 = \delta_{\mathbf{M}}^0 \mu^0 + \delta_{\mathbf{M}}^0 \gamma^0 = 0$ and $g^0 \beta^0 = g^0 \mu^0 + g^0 \gamma^0 = \alpha^0$.

Note that

$$\cdots \rightarrow \text{Hom}_{\mathcal{A}}(D^{-1}, \text{Ker}(g^{-1})) \rightarrow \text{Hom}_{\mathcal{A}}(D^{-1}, \text{Ker}(g^0)) \rightarrow \text{Hom}_{\mathcal{A}}(D^{-1}, \text{Ker}(g^1)) \rightarrow \cdots$$

is also exact. Since $g^0(\beta^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{M}}^{-1} \gamma^{-1}) = \alpha^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{N}}^{-1} g^{-1} \gamma^{-1} = \alpha^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{N}}^{-1} \alpha^{-1} = 0$, $\beta^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{M}}^{-1} \gamma^{-1} \in \text{Hom}_{\mathcal{A}}(D^{-1}, \text{Ker}(g^0))$. Moreover, since $\delta_{\mathbf{M}}^0(\beta^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{M}}^{-1} \gamma^{-1}) =$

$\delta_{\mathbf{M}}^0 \beta^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{M}}^0 \delta_{\mathbf{M}}^{-1} \gamma^{-1} = \delta_{\mathbf{M}}^0 \beta^0 \delta_{\mathbf{D}}^{-1} = 0$, there exists $\mu^{-1} : D^{-1} \rightarrow \text{Ker}(g^{-1})$ such that $\delta_{\mathbf{M}}^{-1} \mu^{-1} = \beta^0 \delta_{\mathbf{D}}^{-1} - \delta_{\mathbf{M}}^{-1} \gamma^{-1}$. Now let $\beta^{-1} = \mu^{-1} + \gamma^{-1}$, and then $\delta_{\mathbf{M}}^{-1} \beta^{-1} = \delta_{\mathbf{M}}^{-1} \mu^{-1} + \delta_{\mathbf{M}}^{-1} \gamma^{-1} = \beta^0 \delta_{\mathbf{D}}^{-1}$ and $g^{-1} \beta^{-1} = g^{-1} \mu^{-1} + g^{-1} \gamma^{-1} = \alpha^{-1}$.

Continuing in this manner, we can get $\beta^k = \mu^k + \gamma^k$ such that $\beta^{k+1} \delta_{\mathbf{D}}^k = \delta_{\mathbf{M}}^k \beta^k$ and $g^k \beta^k = \alpha^k$ for $k = -2, -3, \dots$. Thus, we obtain a morphism $\beta = (\beta^k)_{k \in \mathbb{Z}} : \mathbf{D} \rightarrow \mathbf{M}$ such that $g\beta = \alpha$. This completes the proof. \square

Assume that \mathcal{X} is closed under extensions and has an injective cogenerator. Then according to [20, Lemma 3.3(b)], one has $\text{res } \widehat{\mathcal{X}} \subseteq \text{res } \widetilde{\mathcal{X}}$, that is, every object in \mathcal{A} with finite \mathcal{X} -projective dimension admits a proper \mathcal{X} -resolution. This fact helps us to get the following result, which plays a key role in the proof of Theorem 3.10.

Lemma 3.8. *Assume that \mathcal{X} is closed under extensions and has an injective cogenerator. Then for each object $\mathbf{T} \in \mathbf{K}^-(\text{res } \widehat{\mathcal{X}})$, there exists a short exact sequence*

$$0 \rightarrow \mathbf{K} \rightarrow \mathbf{D} \rightarrow \mathbf{T} \rightarrow 0$$

of complexes such that $\mathbf{D} \in \mathbf{K}^-(\mathcal{X})$, \mathbf{K} is \mathcal{X} -acyclic and it remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$, i.e., the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(X, \mathbf{K}) \rightarrow \text{Hom}_{\mathcal{A}}(X, \mathbf{D}) \rightarrow \text{Hom}_{\mathcal{A}}(X, \mathbf{T}) \rightarrow 0$$

of complexes is exact for each object $X \in \mathcal{X}$.

Proof. Without loss of generality, we may assume

$$\mathbf{T} = \dots \rightarrow T^{-2} \rightarrow T^{-1} \rightarrow T^0 \rightarrow 0 \rightarrow \dots.$$

Let $\mathbf{T}(n) = \dots \rightarrow 0 \rightarrow T^{-n} \rightarrow T^{-n+1} \rightarrow \dots \rightarrow T^0 \rightarrow 0 \rightarrow \dots$ for $n \geq 0$. We will divide the proof into three steps.

Step 1. There exists a short exact sequence $0 \rightarrow \mathbf{K}(0) \rightarrow \mathbf{D}(0) \xrightarrow{\alpha_0} \mathbf{T}(0) \rightarrow 0$ of complexes such that $\mathbf{D}(0) \in \mathbf{K}^-(\mathcal{X})$, $\mathbf{K}(0)$ is \mathcal{X} -acyclic and it remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(X, -)$ for each object $X \in \mathcal{X}$.

Indeed, since $T^0 \in \text{res } \widehat{\mathcal{X}}$ it admits a proper \mathcal{X} -resolution

$$\mathbf{D}(0) = \dots \rightarrow D_0^{-n} \rightarrow \dots \rightarrow D_0^{-1} \rightarrow D_0^0 \rightarrow 0 \rightarrow \dots$$

by [20, Lemma 3.3(b)], i.e., there is an associated exact sequence

$$\mathbf{D}(0)^+ = \dots \rightarrow D_0^{-n} \rightarrow \dots \rightarrow D_0^{-1} \rightarrow D_0^0 \xrightarrow{\alpha_0} T^0 \rightarrow 0.$$

Hence, we obtain the following short exact sequence of complexes

$$\begin{array}{ccccccccc}
& 0 & & 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_0^{-n} & \longrightarrow & \cdots & \longrightarrow & D_0^{-1} & \longrightarrow & \text{Ker}(\alpha_0) & \longrightarrow & 0 & \longrightarrow & \cdots & & \mathbf{K}(0) \\
& \downarrow \parallel & & \downarrow \parallel & & & & \downarrow & & & & \downarrow & & & \downarrow \\
\cdots & \longrightarrow & D_0^{-n} & \longrightarrow & \cdots & \longrightarrow & D_0^{-1} & \longrightarrow & D_0^0 & \longrightarrow & 0 & \longrightarrow & \cdots & & \mathbf{D}(0) \\
& \downarrow & & \downarrow & & & & \downarrow \alpha_0 & & & & \downarrow & & & \downarrow \underline{\alpha}_0 \\
\cdots & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & T^0 & \longrightarrow & 0 & \longrightarrow & \cdots & & \mathbf{T}(0) \\
& \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & & \downarrow \\
& 0 & & 0 & & & & 0 & & & & 0 & & & 0
\end{array} \quad (\#)$$

It is obvious that

$$(\#) \quad 0 \rightarrow \mathbf{K}(0) \rightarrow \mathbf{D}(0) \xrightarrow{\underline{\alpha}_0} \mathbf{T}(0) \rightarrow 0$$

remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(X, -)$ for each object $X \in \mathcal{X}$ and $\mathbf{K}(0)$ is \mathcal{X} -acyclic.

Step 2. For every $n \geq 1$, there exists a short exact sequence

$$0 \rightarrow \mathbf{K}(n) \xrightarrow{\lambda_n} \mathbf{D}(n) \xrightarrow{\underline{\alpha}_n} \mathbf{T}(n) \rightarrow 0 \quad (\natural_n)$$

of complexes such that $\mathbf{D}(n) \in \mathbf{K}^-(\mathcal{X})$, $\mathbf{K}(n)$ is \mathcal{X} -acyclic and it remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(X, -)$ for each object $X \in \mathcal{X}$.

To see this, we may assume that are done for $n - 1$. By a similar argument as in Step 1, one has a short exact sequence

$$0 \rightarrow \mathbf{L} \rightarrow \mathbf{X} \xrightarrow{\beta} T^{-n} \rightarrow 0$$

of complexes, where $\mathbf{L} = \cdots \rightarrow X^{-n} \rightarrow \cdots \rightarrow X^{-1} \rightarrow \text{Ker}(\beta) \rightarrow 0 \rightarrow \cdots$ is \mathcal{X} -acyclic and $\mathbf{X} = \cdots \rightarrow X^{-n} \rightarrow \cdots \rightarrow X^{-1} \rightarrow X^0 \rightarrow 0 \rightarrow \cdots$. Define a morphism $\sigma : T^{-n}[n - 1] \rightarrow \mathbf{T}(n - 1)$ of complexes as follows:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & T^{-n} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow \delta_{\mathbf{T}}^{-n} & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & T^{-n+1} & \xrightarrow{\delta_{\mathbf{T}}^{-n+1}} & T^{-n+2} & \longrightarrow & \cdots & \longrightarrow & T^0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Then it follows from Lemma 3.7 that there exists a morphism $\psi : \mathbf{X}[n-1] \rightarrow \mathbf{D}(n-1)$ of complexes such that the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{L}[n-1] & \longrightarrow & \mathbf{X}[n-1] & \longrightarrow & T^{-n}[n-1] \longrightarrow 0 \\ & & \downarrow \text{dotted} & & \downarrow \psi & & \downarrow \sigma \\ 0 & \longrightarrow & \mathbf{K}(n-1) & \longrightarrow & \mathbf{D}(n-1) & \longrightarrow & \mathbf{T}(n-1) \longrightarrow 0 \end{array}$$

commutes. In particular, the diagram

$$\begin{array}{ccc} X^0 & \xrightarrow{\beta} & T^{-n} \\ \psi^{-n+1} \downarrow & & \downarrow \delta_{\mathbf{T}}^{-n} \\ D_{n-1}^{-n+1} & \xrightarrow{\alpha_{n-1}^{-n+1}} & T^{-n+1} \end{array}$$

commutes.

Let $\mathbf{D}(n)$ be the mapping cone of $\psi : \mathbf{X}[n-1] \rightarrow \mathbf{D}(n-1)$, i.e.,

$$\mathbf{D}(n) = \cdots \rightarrow X^{-1} \oplus D_{n-1}^{-n-1} \xrightarrow{\delta_{\mathbf{D}(n)}^{-n-1}} X^0 \oplus D_{n-1}^{-n} \xrightarrow{\delta_{\mathbf{D}(n)}^{-n}} D_{n-1}^{-n+1} \xrightarrow{\delta_{\mathbf{D}(n-1)}^{-n+1}} D_{n-1}^{-n+2} \rightarrow \cdots$$

in which $\delta_{\mathbf{D}(n)}^{-n} = \begin{pmatrix} 0 & 0 \\ \psi^{-n+1} & \delta_{\mathbf{D}(n-1)}^{-n} \end{pmatrix}$ and $\delta_{\mathbf{D}(n)}^{-n-i+1} = \begin{pmatrix} (-1)^n \delta_{\mathbf{X}}^{-i+2} & 0 \\ \psi^{-n-i+2} & \delta_{\mathbf{D}(n-1)}^{-n-i+1} \end{pmatrix}$ for $i \geq 2$. It is trivial that $\mathbf{D}(n) \in \mathbf{K}^-(\mathcal{X})$ because \mathcal{X} is closed under extensions.

We now define a surjective morphism $\mathbf{D}(n) \rightarrow \mathbf{T}(n)$ of complexes as follows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{-1} \oplus D_{n-1}^{-n-1} & \xrightarrow{\delta_{\mathbf{D}(n)}^{-n-1}} & X^0 \oplus D_{n-1}^{-n} & \xrightarrow{\delta_{\mathbf{D}(n)}^{-n}} & D_{n-1}^{-n+1} \xrightarrow{\delta_{\mathbf{D}(n-1)}^{-n+1}} D_{n-1}^{-n+2} \longrightarrow \cdots \\ & & \downarrow 0 & & \downarrow (\beta, 0) & & \downarrow \alpha_{n-1}^{-n+1} \\ \cdots & \longrightarrow & 0 & \longrightarrow & T^{-n} & \xrightarrow{\delta_{\mathbf{T}}^{-n}} & T^{-n+1} \xrightarrow{\delta_{\mathbf{T}}^{-n+1}} T^{-n+2} \longrightarrow \cdots \end{array}$$

Then we obtain a short exact sequence

$$0 \rightarrow \mathbf{K}(n) \rightarrow \mathbf{D}(n) \rightarrow \mathbf{T}(n) \rightarrow 0,$$

of complexes, which remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(X, -)$ for each $X \in \mathcal{X}$.

Define a morphism $\varphi : \mathbf{K}(n-1) \rightarrow \mathbf{K}(n)$ of complexes as follows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D_{n-1}^{-n-1} & \longrightarrow & D_{n-1}^{-n} & \longrightarrow & \text{Ker}(\alpha_{n-1}^{-n+1}) \longrightarrow \cdots \\ & & \downarrow (0,1) & & \downarrow (0,1) & & \downarrow \parallel \\ \cdots & \longrightarrow & X^{-1} \oplus D_{n-1}^{-n-1} & \longrightarrow & \text{Ker}(\beta) \oplus D_{n-1}^{-n} & \longrightarrow & \text{Ker}(\alpha_{n-1}^{-n+1}) \longrightarrow \cdots \end{array}$$

Then we get a short exact sequence

$$0 \rightarrow \mathbf{K}(n-1) \xrightarrow{\varphi} \mathbf{K}(n) \rightarrow \mathbf{L}[n] \rightarrow 0$$

of complexes, which remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(X, -)$ for each object $X \in \mathcal{X}$. Consequently, $\mathbf{K}(n)$ is also \mathcal{X} -acyclic since $\mathbf{K}(n-1)$ is so by the induction hypothesis.

Step 3. Set

$$\mathbf{D}(n) = \cdots \rightarrow D_n^{-n} \xrightarrow{\delta_{\mathbf{D}(n)}^{-n}} D_n^{-n+1} \rightarrow \cdots \rightarrow D_n^{-1} \xrightarrow{\delta_{\mathbf{D}(n)}^{-1}} D_n^0 \rightarrow 0 \rightarrow \cdots$$

and

$$\mathbf{K}(n) = \cdots \rightarrow K_n^{-n} \xrightarrow{\delta_{\mathbf{K}(n)}^{-n}} K_n^{-n+1} \rightarrow \cdots \rightarrow K_n^{-1} \xrightarrow{\delta_{\mathbf{K}(n)}^{-1}} K_n^0 \rightarrow 0 \rightarrow \cdots$$

for each $n \geq 0$. Note that $D_n^{-n+1} = D_{n-1}^{-n+1}$ and $K_n^{-n+1} = K_{n-1}^{-n+1}$ for all $n \geq 1$. So we have the following complexes

$$\mathbf{D} = \cdots \rightarrow D_n^{-n} \xrightarrow{\delta_{\mathbf{D}}^{-n}} D_{n-1}^{-n+1} \rightarrow \cdots \rightarrow D_1^{-1} \xrightarrow{\delta_{\mathbf{D}}^{-1}} D_0^0 \rightarrow 0 \rightarrow \cdots$$

and

$$\mathbf{K} = \cdots \rightarrow K_n^{-n} \xrightarrow{\delta_{\mathbf{K}}^{-n}} K_{n-1}^{-n+1} \rightarrow \cdots \rightarrow K_1^{-1} \xrightarrow{\delta_{\mathbf{K}}^{-1}} K_0^0 \rightarrow 0 \rightarrow \cdots$$

where $\delta_{\mathbf{D}}^{-n} = \delta_{\mathbf{D}(n)}^{-n}$ and $\delta_{\mathbf{K}}^{-n} = \delta_{\mathbf{K}(n)}^{-n}$ for each $n \geq 0$. In addition, there exists a short exact sequence

$$0 \rightarrow \mathbf{K} \xrightarrow{\lambda} \mathbf{D} \xrightarrow{\alpha} \mathbf{T} \rightarrow 0$$

of complexes, where $\lambda^{-n} = \lambda_n^{-n} : K_n^{-n} \rightarrow D_n^{-n}$ and $\alpha^{-n} = \alpha_n^{-n} : D_n^{-n} \rightarrow T_n^{-n}$ for each $n \geq 0$ (see Step 2).

It is easy to see that $0 \rightarrow \mathbf{K} \xrightarrow{\lambda} \mathbf{D} \xrightarrow{\alpha} \mathbf{T} \rightarrow 0$ is as desired. This completes the proof. \square

From [13, Theorem 4.6(1)] we know that $\text{res } \tilde{\mathcal{X}}$ is closed under direct summands. This fact enables us to get the next result.

Lemma 3.9. *Assume that \mathcal{X} is exact and has an injective cogenerator. Then $\text{res } \hat{\mathcal{X}}$ is closed under direct summands.*

Proof. Let M be an object in \mathcal{A} such that $M \in \text{res } \hat{\mathcal{X}}$ and M' any direct summand of M . It suffices to prove that $M' \in \text{res } \hat{\mathcal{X}}$. Note that M is also in $\text{res } \tilde{\mathcal{X}}$ (see [20, Lemma 3.3(b)]). It follows from [13, Theorem 4.6(1)] that $M' \in \text{res } \tilde{\mathcal{X}}$. Since \mathcal{X} is closed under extensions and has an injective cogenerator, $\text{Ext}_{\mathcal{X}\mathcal{A}}^n(M, -) = 0$ for $n > \mathcal{X}\text{-pd}(M)$ by [20, Proposition 4.5(b)(2)]. Hence, $\text{Ext}_{\mathcal{X}\mathcal{A}}^n(M', -) = 0$ for $n > \mathcal{X}\text{-pd}(M)$, and so $M' \in \text{res } \hat{\mathcal{X}}$ by [20, Proposition 4.5(a)] since \mathcal{X} is also closed under direct summands, as desired. \square

Now let $H : \mathbf{K}^-(\mathcal{X}) \rightarrow \mathbf{D}_{\mathcal{X}}^-(\text{res } \widehat{\mathcal{X}})$ be the composition of the embedding $\mathbf{K}^-(\mathcal{X}) \hookrightarrow \mathbf{K}^-(\text{res } \widehat{\mathcal{X}})$ and the localization $Q : \mathbf{K}^-(\text{res } \widehat{\mathcal{X}}) \rightarrow \mathbf{D}_{\mathcal{X}}^-(\text{res } \widehat{\mathcal{X}})$. Clearly, H is a triangle functor. Moreover, Lemma 3.6 implies that H is fully faithful (the case $\mathcal{S} = \text{res } \widehat{\mathcal{X}}$), and from Lemma 3.8 we know that H is dense. Thus, we obtain the main result of this paper.

Theorem 3.10. *Assume that \mathcal{X} is exact and has an injective cogenerator. Then there exists a triangle-equivalence $\mathbf{K}^-(\mathcal{X}) \cong \mathbf{D}_{\mathcal{X}}^-(\text{res } \widehat{\mathcal{X}})$.*

In [5, Section 3] Chen investigated the relative derived category of the case $\mathcal{S} = \mathcal{A}$ in Definition 3.4. Let $\mathcal{X} \subseteq \mathcal{A}$ be a contravariantly finite subcategory. Assume that \mathcal{X} is admissible and $\mathcal{X}\text{-pd}(\mathcal{A}) < \infty$. Chen proved in [5, Proposition 3.5] that the natural composite functor $\mathbf{K}(\mathcal{X}) \hookrightarrow \mathbf{K}(\mathcal{A}) \xrightarrow{Q} \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ is a triangle-equivalence.

In view of the following example, we can conclude that Theorem 3.10 can not be deduced from [5, Proposition 3.5].

Example 3.11. *Let $R = \mathbb{Z}_4$ and $\mathcal{X} = \mathcal{P}_R$, where \mathcal{P}_R denotes the class of all projective R -modules. It is clear that \mathcal{X} is exact with itself as an injective cogenerator. Then it follows from Theorem 3.10 that there exists a triangle-equivalence $\mathbf{K}^-(\mathcal{X}) \cong \mathbf{D}_{\mathcal{X}}^-(\text{res } \widehat{\mathcal{X}})$. But here $\text{res } \widehat{\mathcal{X}}$ is not an abelian category since the kernel of the morphism $f : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$ with $f(x) = 2x$ is $2\mathbb{Z}_4$ and $2\mathbb{Z}_4$ has infinite projective dimension.*

Recall from [21, Definition 4.1] that an acyclic complex of objects in \mathcal{X} is called *totally \mathcal{X} -acyclic* if it is $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{X})$ -exact. Let $\mathcal{G}(\mathcal{X})$ denote the subcategory of \mathcal{A} whose modules are of the form $M \cong \text{Ker}(\delta_{\mathbf{X}}^1)$ for some totally \mathcal{X} -acyclic complex \mathbf{X} . In the special case $\mathcal{X} = \mathcal{P}$, the objects in $\mathcal{G}(\mathcal{P})$ are called Gorenstein projective objects of \mathcal{A} .

Remark 3.12. *Assume $\mathcal{X} \perp \mathcal{X}$. Then by virtue of [21, Theorem B], $\mathcal{G}(\mathcal{X})$ is an exact subcategory of \mathcal{A} , and it is closed under kernels of epimorphisms if \mathcal{X} is so. Moreover, it follows from [21, Corollary 4.7] that \mathcal{X} is both an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{X})$,*

According to Remark 3.12, we can obtain the next result by Theorem 3.10.

Corollary 3.13. *Assume $\mathcal{X} \perp \mathcal{X}$. Then there exists a triangle-equivalence $\mathbf{K}^-(\mathcal{G}(\mathcal{X})) \cong \mathbf{D}_{\mathcal{G}(\mathcal{X})}^-(\text{res } \widehat{\mathcal{G}(\mathcal{X})})$.*

It is obvious that $\mathcal{X} = \mathcal{P}$ satisfies the hypothesis of Corollary 3.13. Hence, we have the next triangle-equivalence, which extends [10, Theorem 3.6(1)] to the bounded below case.

Corollary 3.14. $\mathbf{K}^-(\mathcal{G}(\mathcal{P})) \cong \mathbf{D}_{\widehat{\mathcal{G}(\mathcal{P})}}^-(\text{res } \widehat{\mathcal{G}(\mathcal{P})})$.

The notion of semidualizing module goes back to [8] (where the more general PG-module are studied) and [9]. Christensen [3] extended this notion to semidualizing complex. Recently, it has already been defined over arbitrary associative rings [12, Definition 2.1]. In what follows we assume that R is a commutative ring. Recall from [24] that an R -module C is called *semidualizing* if C admits a degree-wise finite projective resolution, $\text{Ext}_R^{\geq 1}(C, C) = 0$, and the natural homothety map $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism. More examples of semidualizing module can be found in [3, 12, 14].

Following [24], an R -module is called *C -projective* (resp., *C -flat*) if it is isomorphic to an R -module of the form $C \otimes_R P$ for some projective (resp., flat) R -module P . An R -module is called *C -injective* if it is isomorphic to an R -module of the form $\text{Hom}_R(C, I)$ for some injective R -module I . Let \mathcal{P}_C (resp., \mathcal{F}_C , \mathcal{I}_C) denote the subcategory of C -projective (resp., C -flat, C -injective) R -modules. A *complete \mathcal{PP}_C -resolution* is an exact and $\text{Hom}_R(-, \mathcal{P}_C)$ -exact complex \mathbf{X} of R -modules with X_i projective for $i \leq 0$ and X_i C -projective for $i > 0$. An R -module M is \mathcal{G}_C -*projective* if there exists a complete \mathcal{PP}_C -resolution \mathbf{X} such that $M \cong \text{Ker}(\delta_{\mathbf{X}}^1)$. A *complete \mathcal{FF}_C -resolution* is an exact and $- \otimes_R \mathcal{I}_C$ -exact complex \mathbf{Z} of R -modules with Z_i flat for $i \leq 0$ and Z_i C -flat for $i > 0$. An R -module T is \mathcal{G}_C -*flat* if there exists a complete \mathcal{FF}_C -resolution \mathbf{Z} such that $T \cong \text{Ker}(\delta_{\mathbf{Z}}^1)$. Let \mathcal{GP}_C (resp., \mathcal{GF}_C) denote the subcategory of \mathcal{G}_C -projective (resp., \mathcal{G}_C -flat) R -modules.

For more details about semidualizing modules and their related categories, we refer the reader to [12, 14, 19-24].

Corollary 3.15. *Let R be a commutative ring and C a semidualizing R -module. Then there exists a triangle-equivalence $\mathbf{K}^-(\mathcal{GP}_C) \cong \mathbf{D}_{\widehat{\mathcal{GP}_C}}^-(\text{res } \widehat{\mathcal{GP}_C})$.*

Proof. From [24, Proposition 3.15] we know that \mathcal{GP}_C is exact. Moreover, according to [24, Proposition 2.6 and 2.7], $\mathcal{P}_C \subseteq \mathcal{GP}_C$ and $\mathcal{GP}_C \perp \mathcal{P}_C$. Hence, it follows from [24, Proposition 3.16] that \mathcal{P}_C is an injective cogenerator for \mathcal{GP}_C . Then the result follows by Theorem 3.10. \square

If R is commutative noetherian, [19, Theorem I] said that \mathcal{GF}_C is exact and $\mathcal{F}_C^{\text{cot}}$ is an injective cogenerator for \mathcal{GF}_C , where $\mathcal{F}_C^{\text{cot}}$ denotes the subcategory of R -modules $C \otimes_R H$ with H flat and cotorsion. Then by Theorem 3.10, we have

Corollary 3.16. *Let R be a commutative noetherian ring and C a semidualizing R -module. Then there exists a triangle-equivalence $\mathbf{K}^-(\mathcal{GF}_C) \cong \mathbf{D}_{\widehat{\mathcal{GF}_C}}^-(\text{res } \widehat{\mathcal{GF}_C})$.*

4. Relative Cohomology and Applications

We begin this section with the following result, which will be used in the proofs of Theorem 4.2 and Proposition 4.5.

Proposition 4.1. $\mathbf{D}_{\mathcal{X}}^{-}(\mathcal{A})$ is a full triangulated subcategory of $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$. $\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})$ is a full triangulated subcategory of $\mathbf{D}_{\mathcal{X}}^{-}(\mathcal{A})$, and hence of $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$.

Proof. We only prove the first assertion, and the second one can be proved similarly. Let $\Sigma_{\mathcal{X}}$ be the compatible multiplicative system determined by $\mathbf{K}_{\mathcal{X}}(\mathcal{A})$, that is, the collection of all \mathcal{X} -quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$. Then $\mathbf{D}_{\mathcal{X}}(\mathcal{A}) = \Sigma_{\mathcal{X}}^{-1}\mathbf{K}(\mathcal{A})$ and $\mathbf{D}_{\mathcal{X}}^{-}(\mathcal{A}) = (\Sigma_{\mathcal{X}} \cap \mathbf{K}^{-}(\mathcal{A}))^{-1}\mathbf{K}^{-}(\mathcal{A})$. By [11, Proposition (III) 2.10] it suffices to prove that for any \mathcal{X} -quasi-isomorphism $f : \mathbf{X} \rightarrow \mathbf{Y}$ with $\mathbf{Y} \in \mathbf{K}^{-}(\mathcal{A})$, there is a morphism $g : \mathbf{X}' \rightarrow \mathbf{X}$ with $\mathbf{X}' \in \mathbf{K}^{-}(\mathcal{A})$ such that fg is an \mathcal{X} -quasi-isomorphism. Then the canonical functor $(\Sigma_{\mathcal{X}} \cap \mathbf{K}^{-}(\mathcal{A}))^{-1}\mathbf{K}^{-}(\mathcal{A}) \rightarrow \Sigma_{\mathcal{X}}^{-1}\mathbf{K}(\mathcal{A})$ is fully faithful, and hence $\mathbf{D}_{\mathcal{X}}^{-}(\mathcal{A})$ is a full triangulated subcategory of $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$.

Suppose that there is an integer i such that $Y^k = 0$ for any $k > i$. Let \mathbf{X}' be the soft truncation $\mathbf{X}_{i\supset}$ of \mathbf{X} . Then there is a commutative diagram

$$\begin{array}{ccccccccc}
 \mathbf{X}_{i\supset} & \cdots & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & \text{Ker}(\delta_{\mathbf{X}}^i) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 g \downarrow & & & \parallel & & \parallel & & \downarrow & & \downarrow & & \\
 \mathbf{X} & \cdots & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \\
 f \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \mathbf{Y} & \cdots & \longrightarrow & Y^{i-2} & \longrightarrow & Y^{i-1} & \longrightarrow & Y^i & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

It is easy to check that g is also an \mathcal{X} -quasi-isomorphism, and so is fg . This completes the proof. \square

Now we are in a position to give our another main result, which is an analog of [10, Theorem 3.12].

Theorem 4.2. Let M and N be objects in \mathcal{A} . Assume that $M \in \text{res } \tilde{\mathcal{X}}$. Then $\text{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N) = \text{Hom}_{\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})}(M, N[n])$.

Proof. Let $\mathbf{X} \xrightarrow{\cong} M$ be a proper \mathcal{X} -resolution of M . Then $\mathbf{X} \xrightarrow{\cong} M$ is an \mathcal{X} -quasi-isomorphism, and so $\mathbf{X} \cong M$ in $\mathbf{D}_{\mathcal{X}}^{-}(\mathcal{A})$. This yields the third isomorphism in

the following sequence

$$\begin{aligned}
\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N) &= H^n(\mathrm{Hom}_{\mathcal{A}}(\mathbf{X}, N)) \\
&\cong \mathrm{Hom}_{\mathbf{K}^-(\mathcal{A})}(\mathbf{X}, N[n]) \\
&\cong \mathrm{Hom}_{\mathbf{D}_{\tilde{\mathcal{X}}}^-(\mathcal{A})}(\mathbf{X}, N[n]) \\
&\cong \mathrm{Hom}_{\mathbf{D}_{\tilde{\mathcal{X}}}^-(\mathcal{A})}(M, N[n]) \\
&\cong \mathrm{Hom}_{\mathbf{D}_{\tilde{\mathcal{X}}}^b(\mathcal{A})}(M, N[n]),
\end{aligned}$$

while the second isomorphism follows from Lemma 3.6 (the case $\mathcal{S} = \mathcal{A}$), and the fourth one holds by Proposition 4.1. This completes the proof. \square

The next result follows directly from [20, Lemma 3.3(b)] and Theorem 4.2. It will be used in the proof of Proposition 4.5.

Corollary 4.3. *Let M be an object in \mathcal{A} such that $M \in \mathrm{res} \hat{\mathcal{X}}$. Assume that \mathcal{X} is closed under extensions and has an injective cogenerator. Then $\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N) = \mathrm{Hom}_{\mathbf{D}_{\tilde{\mathcal{X}}}^b(\mathcal{A})}(M, N[n])$.*

In the remainder of the paper, we display two applications of Theorem 4.2. Firstly, we give in the relative derived category a brief proof for the results of long exact sequences about relative derived functor $\mathrm{Ext}_{\mathcal{X}\mathcal{A}}(-, -)$. For the similar results with different methods, we refer the reader to [20, Lemma 4.3] or [6, Lemma 8.2.3 and 8.2.5].

Proposition 4.4. *Let M be an object in \mathcal{A} and $N^\bullet = 0 \rightarrow N \xrightarrow{f} N' \xrightarrow{g} N'' \rightarrow 0$ an \mathcal{X} -acyclic sequence in \mathcal{A} .*

(1) *If $M \in \mathrm{res} \tilde{\mathcal{X}}$, then there exist homomorphisms $\vartheta_{\mathcal{X}}^n(M, N^\bullet)$, which are natural in M and N^\bullet , such that the sequence below is exact*

$$\begin{aligned}
\cdots &\longrightarrow \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N) \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, f)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N') \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, g)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(M, N'') \\
&\xrightarrow{\vartheta_{\mathcal{X}}^n(M, N^\bullet)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(M, N) \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(M, f)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(M, N') \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(M, g)} \cdots
\end{aligned}$$

(2) *If N, N' and $N'' \in \mathrm{res} \tilde{\mathcal{X}}$, then there exist homomorphisms $\vartheta_{\mathcal{X}}^n(N^\bullet, M)$, which are natural in N^\bullet and M , such that the sequence below is exact*

$$\begin{aligned}
\cdots &\longrightarrow \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(N'', M) \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(g, M)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(N', M) \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(f, M)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^n(N, M) \\
&\xrightarrow{\vartheta_{\mathcal{X}}^n(N^\bullet, M)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(N'', M) \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(g, M)} \mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(N', M) \xrightarrow{\mathrm{Ext}_{\mathcal{X}\mathcal{A}}^{n+1}(f, M)} \cdots
\end{aligned}$$

Proof. Consider all objects M, N, N' and N'' being complexes concentrated in 0th degree, and f, g being morphisms of such complexes. Since N^\bullet is \mathcal{X} -acyclic, g induces an \mathcal{X} -quasi-isomorphism $\text{cone}(f) \rightarrow N''$ in $\mathbf{K}^b(\mathcal{A})$, where the mapping $\text{cone}(f) = \cdots \rightarrow 0 \rightarrow N \xrightarrow{f} N' \rightarrow 0 \rightarrow \cdots$. Hence, $\text{cone}(f) \cong N''$ in $\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})$.

Consider the following commutative diagram in $\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})$

$$\begin{array}{ccccccc} N & \xrightarrow{f} & N' & \longrightarrow & \text{cone}(f) & \longrightarrow & N[1] \\ \parallel & & \parallel & & \downarrow & & \parallel \\ N & \xrightarrow{f} & N' & \xrightarrow{g} & N'' & \longrightarrow & N[1]. \end{array}$$

This implies that $N \xrightarrow{f} N' \xrightarrow{g} N'' \rightarrow N[1]$ is a distinguished triangle in $\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})$. Applying the functors $\text{Hom}_{\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})}(M, -)$ and $\text{Hom}_{\mathbf{D}_{\mathcal{X}}^b(\mathcal{A})}(-, M)$ respectively to this distinguished triangle, and by Theorem 4.2, we get our desired long exact sequences. \square

In what follows, we pay our attention to ${}_R\mathcal{M}$ (the category of left R -modules), and then let \mathcal{X} and \mathcal{W} denote subcategories of ${}_R\mathcal{M}$.

Recall from [22, Definition 3.1] that a *Tate \mathcal{W} -resolution* of an R -module M is a diagram $\mathbf{T} \xrightarrow{\nu} \mathbf{W} \xrightarrow{\pi} M$ of morphisms of complexes, where $\mathbf{W} \xrightarrow{\pi} M$ is a proper \mathcal{W} -resolution, \mathbf{T} is a totally \mathcal{W} -acyclic complex, and ν_i is bijective for all $i \ll 0$. For any R -module N and each $n \in \mathbb{Z}$, the *n th Tate cohomology group* is defined as

$$\widehat{\text{Ext}}_{\mathcal{W}\mathcal{M}}^n(M, N) := H^n(\text{Hom}_R(\mathbf{T}, N)),$$

which is independent of choices of resolutions and lifting (see [22, Definition 4.1 and Lemma 3.8]).

In [15] Liang and Yang gave us another way to calculate such Tate cohomology group. Let M be an object in $\text{res } \widehat{\mathcal{X}}$. If \mathcal{X} is exact and closed under kernels of epimorphisms, and \mathcal{W} is both an injective cogenerator and a projective generator for \mathcal{X} and closed under direct summands, then [22, Lemma 3.4] implies that M admits a Tate \mathcal{W} -resolution $\mathbf{T} \xrightarrow{\nu} \mathbf{W} \xrightarrow{\pi} M$. Moreover, M admits a proper \mathcal{X} -resolution $\mathbf{X} \xrightarrow{\cong} M$ by [20, Lemma 3.3(b)]. Let $f : \mathbf{W} \rightarrow \mathbf{X}$ be a lifting of the identity $\text{Id}_M : M \rightarrow M$. Liang and Yang argued that, for $n \geq 1$, $\widehat{\text{Ext}}_{\mathcal{W}\mathcal{M}}^n(M, N)$ is exactly $H^{n+1}(\text{Hom}_R(\text{cone}(f), N))$ (see [15, Proposition 3.13]). Hence, $\widehat{\text{Ext}}_{\mathcal{W}\mathcal{M}}^n(M, N) \cong \text{Hom}_{\mathbf{K}^-(R)}(\text{cone}(f)[-n-1], N)$ for $n \geq 1$.

Next, we give another application of Theorem 4.2. Note that the following Avramov-Martinskivsky type exact sequence follows directly from [22, Theorem 4.10]. But here we give also a proof in the relative derived category.

Proposition 4.5. *Assume that \mathcal{X} is exact and closed under kernels of epimorphisms, and \mathcal{W} is both an injective cogenerator and a projective generator for \mathcal{X} and closed under direct summands. Let M and N be R -modules with $M \in \text{res } \widehat{\mathcal{X}}$. Set $d = \mathcal{X}\text{-pd}(M)$. Then there exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow \text{Ext}_{\mathcal{X}\mathcal{M}}^1(M, N) &\longrightarrow \text{Ext}_{\mathcal{W}\mathcal{M}}^1(M, N) \longrightarrow \widehat{\text{Ext}}_{\mathcal{W}\mathcal{M}}^1(M, N) \\ &\longrightarrow \text{Ext}_{\mathcal{X}\mathcal{M}}^2(M, N) \longrightarrow \text{Ext}_{\mathcal{W}\mathcal{M}}^2(M, N) \longrightarrow \widehat{\text{Ext}}_{\mathcal{W}\mathcal{M}}^2(M, N) \longrightarrow \\ \cdots \longrightarrow \text{Ext}_{\mathcal{X}\mathcal{M}}^d(M, N) &\longrightarrow \text{Ext}_{\mathcal{W}\mathcal{M}}^d(M, N) \longrightarrow \widehat{\text{Ext}}_{\mathcal{W}\mathcal{M}}^d(M, N) \longrightarrow 0. \end{aligned}$$

Proof. For $f : \mathbf{W} \rightarrow \mathbf{X}$, there is a distinguished triangle

$$\mathbf{W} \xrightarrow{f} \mathbf{X} \longrightarrow \text{cone}(f) \longrightarrow \mathbf{W}[1]$$

in $\mathbf{K}^-(R)$. By applying the functor $\text{Hom}_{\mathbf{K}^-(R)}(-, N)$ to it, we then get an exact sequence

$$\begin{aligned} \cdots \longrightarrow \text{Hom}_{\mathbf{K}^-(R)}(\mathbf{W}[-n+1], N) &\longrightarrow \text{Hom}_{\mathbf{K}^-(R)}(\text{cone}(f)[-n], N) \\ &\longrightarrow \text{Hom}_{\mathbf{K}^-(R)}(\mathbf{X}[-n], N) \longrightarrow \text{Hom}_{\mathbf{K}^-(R)}(\mathbf{W}[-n], N) \\ \longrightarrow \text{Hom}_{\mathbf{K}^-(R)}(\text{cone}(f)[-n-1], N) &\longrightarrow \text{Hom}_{\mathbf{K}^-(R)}(\mathbf{X}[-n-1], N) \longrightarrow \cdots \end{aligned}$$

As $\mathbf{X} \xrightarrow{\cong} M$ is an \mathcal{X} -quasi-isomorphism, $\mathbf{X} \cong M$ in $\mathbf{D}_{\mathcal{X}}^-(R)$. This yields the third isomorphism in the following sequence

$$\begin{aligned} \text{Hom}_{\mathbf{K}^-(R)}(\mathbf{X}[-n], N) &\cong \text{Hom}_{\mathbf{K}^-(R)}(\mathbf{X}, N[n]) \\ &\cong \text{Hom}_{\mathbf{D}_{\mathcal{X}}^-(R)}(\mathbf{X}, N[n]) \\ &\cong \text{Hom}_{\mathbf{D}_{\mathcal{X}}^-(R)}(M, N[n]) \\ &\cong \text{Hom}_{\mathbf{D}_{\mathcal{X}}^b(R)}(M, N[n]) \\ &\cong \text{Ext}_{\mathcal{X}\mathcal{M}}^n(M, N), \end{aligned}$$

while the second isomorphism follows from Lemma 3.6 (the case $\mathcal{S} = R$), the fourth isomorphism is from Proposition 4.1, and the last one holds by Corollary 4.3. Similarly, we have $\text{Hom}_{\mathbf{K}^-(R)}(\mathbf{W}[-n], N) \cong \text{Ext}_{\mathcal{W}\mathcal{M}}^n(M, N)$. By left exactness of Hom , it is easy to check that $\text{Hom}_{\mathbf{K}^-(R)}(\text{cone}(f)[-1], N) \cong H^1(\text{Hom}_R(\text{cone}(f), N)) = 0$. Moreover, $\text{Ext}_{\mathcal{X}\mathcal{M}}^{d+1}(M, N)$ vanishes by [20, Lemma 4.5.(b)(2)]. Thus, we have the desired exact sequence. \square

According to [22, Fact 2.6], we know that $\mathcal{P}_C \perp \mathcal{P}_C$, and \mathcal{P}_C is exact and closed under kernels of epimorphisms. Hence, it follows from Remark 3.12 that $\mathcal{G}(\mathcal{P}_C)$ is also exact and closed under kernels of epimorphisms, and \mathcal{P}_C is both an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{P}_C)$. Then we have the following result by Proposition 4.5.

Corollary 4.6. [22, Theorem B] *Let R be a commutative ring and C a semidualizing R -module. Assume that M and N are R -modules with $M \in \text{res } \widehat{\mathcal{G}(\mathcal{P}_C)}$. Set $d = \mathcal{G}(\mathcal{P}_C)\text{-pd}(M)$. Then there exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow \text{Ext}_{\mathcal{G}(\mathcal{P}_C)\mathcal{M}}^1(M, N) &\longrightarrow \text{Ext}_{\mathcal{P}_C\mathcal{M}}^1(M, N) \longrightarrow \widehat{\text{Ext}}_{\mathcal{P}_C\mathcal{M}}^1(M, N) \\ &\longrightarrow \text{Ext}_{\mathcal{G}(\mathcal{P}_C)\mathcal{M}}^2(M, N) \longrightarrow \text{Ext}_{\mathcal{P}_C\mathcal{M}}^2(M, N) \longrightarrow \widehat{\text{Ext}}_{\mathcal{P}_C\mathcal{M}}^2(M, N) \longrightarrow \\ \cdots \longrightarrow \text{Ext}_{\mathcal{G}(\mathcal{P}_C)\mathcal{M}}^d(M, N) &\longrightarrow \text{Ext}_{\mathcal{P}_C\mathcal{M}}^d(M, N) \longrightarrow \widehat{\text{Ext}}_{\mathcal{P}_C\mathcal{M}}^d(M, N) \longrightarrow 0. \end{aligned}$$

Remark 4.7. *Under appropriate dual hypotheses, all the results in this paper have their dual versions.*

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